Math 213 Exam 1, 2024 Spring Answer Key

15 points 1.) Let

$\vec{u} = \langle 3, -1, 0 \rangle = 3\vec{i} - \vec{j}$,

$\vec{v} = \langle 4, 4, -2 \rangle = 4\vec{i} + 4\vec{j} - 2\vec{k}$

a.) find $\hat{u} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{\langle 3, -1, 0 \rangle}{\sqrt{(3)^2 + (-1)^2 + (0)^2}} = \boxed{\left\langle \dfrac{3}{\sqrt{10}}, \dfrac{-1}{\sqrt{10}}, 0 \right\rangle}$

b.) find $\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$

$$= \frac{(3(4) - 1(4) + 0(-2))}{4^2 + 4^2 + (-2)^2} \langle 4, 4, -2 \rangle$$

$$= \frac{8}{36} \langle 4, 4, -2 \rangle = \boxed{\left\langle \frac{8}{9}, \frac{8}{9}, -\frac{4}{9} \right\rangle}$$

$\frac{2}{9}$

c.) write $\vec{u}$ as a linear combination of a vector parallel to $\vec{v}$ and a vector perpendicular to $\vec{v}$

$\vec{u} = \underbrace{\text{vector} \parallel \text{to } \vec{v}}_{\text{projection}} + \text{vector} \perp \text{to } \vec{v}$

projection $+ (\vec{u} - \text{proj}_{\vec{v}}\vec{u})$

$$\left\langle \frac{8}{9}, \frac{8}{9}, \frac{-4}{9} \right\rangle + \left(\langle 3, -1, 0 \rangle - \left\langle \frac{8}{9}, \frac{8}{9}, \frac{-4}{9} \right\rangle \right)$$

$$= \boxed{\left\langle \frac{8}{9}, \frac{8}{9}, \frac{-4}{9} \right\rangle + \left\langle \frac{19}{9}, \frac{-17}{9}, \frac{4}{9} \right\rangle}$$

• check dot product $= 0$; $\dfrac{1}{81} \langle 8 \cdot 19 - 8 \cdot 7 - 4 \cdot 4 \rangle = \dfrac{0}{81} = 0$

10 pts 2.) Consider the planes:

$$x + 2y + z = 1 \qquad x - y + 2z = -7$$

• Find a parametric equation for the line of intersection of the two planes.

• Let $z = 0$; $x + 2y = 1$ & $x - y = -7$

subtract : $\begin{array}{r} x + 2y = 1 \\ -\underline{\;x + y = +7\;} \\ 3y = 8 \end{array}$ $\quad y = \frac{8}{3}$

plug in y : $x - 2y = 1 - 2\left(\frac{8}{3}\right) = 1 - \frac{16}{3} = \frac{-13}{3}$

so $\left(\frac{-13}{3}, \frac{8}{3}, 0\right)$ is on both lines

• A vector in the direction of the line is

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = i5 - j - 3k$$

line of intersection $\boxed{\left\langle -\frac{13}{3} + 5t, \frac{8}{3} - t, -3t \right\rangle}$

or $\boxed{x = -\frac{13}{3} + 5t, \; y = \frac{8}{3} - t, \; z = -3t}$

pts 3.) consider the curve $\vec{r}(t) = \frac{1}{3}t^3\vec{i} + \frac{1}{2}t^2\vec{j}$, where $t \geq 0$.

a.) find the length of the curve for values of $0 \leq t \leq 1$

$$r'(t) = t^2\vec{i} + t\vec{j}$$

length: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$$= \int_0^1 \sqrt{(t^2)^2 + (t)^2}\, dt = \int_0^1 t\sqrt{t^2+1}\, dt$$

let $u = t^2+1$ $\Rightarrow \int_0^1 t\sqrt{t^2+1}\, dt$

$du = 2t\,dt$

$$= \frac{1}{2}\int_1^2 \sqrt{u}\, du$$

@ $t=1$; $u = 1^2+1 = 2$

@ $t=0$; $u = 0^2+1 = 1$ $\quad = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{\frac{3}{2}} \Big|_1^2$

$$= \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right)$$

find the

b.) unit tangent vector to the curve at the point (4, 2).

unit vector of a smooth curve: $T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt} = \frac{v}{|v|}$

$v = r'(t) = t^2\vec{i} + t\vec{j}$ $\quad$ solve $\frac{1}{3}t^3 = 4$, $\frac{1}{2}t^2 = 2 \Rightarrow t^2 = 4 \Rightarrow t = 2$

$v = r'(2)$ $\quad$ since $t \geq 0$

$$\frac{1}{3}(2)^3 = 4 \Rightarrow \frac{8}{3} \neq 4$$

The point (4,2) is not on the curve so there is no tangent vector at that point.

$$\frac{4i+2j}{\sqrt{4^2+2^2}} = \frac{4}{2\sqrt{5}}i + \frac{2}{2\sqrt{5}}j$$

$$= \frac{1}{\sqrt{5}}\left(2\vec{i} + \vec{j}\right)$$

However if the point was $\left(\frac{8}{3}, 2\right)$ you get $r'(2) = 4\vec{i} + 2\vec{j}$ & the tangent vector is

Let
10 pts 4.) $\vec{u} = \vec{i} + \vec{j} - k$, $\vec{v} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{\omega} = -\vec{i} - 2\vec{j} + 3\vec{k}$.

a.) find the area of the parallelogram with sides given by the vectors $\vec{u}$ and $\vec{v}$

$$\left\| \begin{matrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{matrix} \right\| = |2\vec{i} - 3\vec{j} - \vec{k}| = \sqrt{2^2 + (-3)^2 (-1)^2} = \sqrt{14}$$

cubic units

$|u \times v|$
area of parallelogram

b.) find the volume of the parallelopiped

volume of the parallelopiped: $|(u \times v) \cdot \omega|$

$|\langle 2, -3, -1 \rangle \cdot \langle -1, -2, 3 \rangle| = |-2 + 6 - 3| = 1$ cubic units

10 pts 5.) a.) identify the following quadratic surface and state its center (x_0, y_0, z_0): $x^2 + y^2 + z^2 - 6y + 8z = 0$

$\Rightarrow \quad x^2 + (y^2 - 6y) + (z^2 + 8z) = 0$

$x^2 + (y-3)^2 - 9 + (z+4)^2 - 16 = 0$ ← complete the square

$x^2 + (y-3)^2 + (z+4)^2 = 25$

center: $(0, 3, -4)$; circle of radius: 5

b.) identify the following quadratic surface and state the line corresponding to its axis of symmetry : $x = y^2 + z^2$

← elliptic paraboloid

- cross sections perpendicular to the x axis are circles

- axis of symmetry is the x-axis $\vec{r}(t) = \langle t, 0, 0 \rangle$

15 pts (6.) Consider the following line ℓ and plane P:

$\ell : \vec{r}(t) = (2,2,1)t + (3,0,0)$ P: $x+3y-z=-4$

a.) find the point P_0 where ℓ intersects P.

$\vec{r}(t) = \langle 2t+3, 2t, t \rangle$ intersects the plane

$x + 3y - z = -4$

$\Rightarrow (2t+3) + 3(2t) - t = -4 \Rightarrow 2t+3+6t-t=-4$

$\Rightarrow 7t = -7 \Rightarrow t = -1$

intersection is at $\langle 2(-1)+3, 2(-1), -1 \rangle = (1,-2,-1)$

b.) find the line that intersects P_0 and is normal to P

$\vec{n} = \langle 1, 3, 1 \rangle$

$\vec{r}(t) = \vec{n}t + \vec{r_0}$

$\vec{r}(t) = \langle 1, 3, -1 \rangle t + \langle 1, -2, -1 \rangle$

or

$x = 1+t,\ y = -2+3t,\ z = -1-t$

10pts 7.) find the distance from the point $(6,1,0)$ to the plane $x-2y+z=0$

- a pt on the plane is $S(0,0,0)$

$\Rightarrow \vec{PS} = \langle 0-6, 0-1, 0\rangle = \langle -6,-1,0\rangle$

$\vec{n} = \langle 1,-1,1\rangle$

distance $= |PS|\cos\theta$

$= \dfrac{|PS|\,|n|\cos\theta}{|n|} = \left|\dfrac{PS\cdot n}{|n|}\right|$

$= \left|\dfrac{(-6)(1)+(-1)(-1)+(0)(1)}{\sqrt{1^2+(-1)^2+1^2}}\right| = \left|-\dfrac{5}{\sqrt{3}}\right| = \boxed{\dfrac{5}{\sqrt{3}}}$

8.) 15 pts suppose that a particle is constrained to a wire and follows the path $\vec{r}(t) = \cos(t)\vec{i} + 5\sin(t)\vec{j}$

a.) find the velocity and acceleration of the particle at $t = \frac{\pi}{4}$

$v(t) = r'(t)$; $a(t) = v'(t) = r''(t)$

$r'(t) = -\sin t\,\vec{i} + 5\cos t\,\vec{j}$

$r'(\frac{\pi}{4}) = -\sin(\frac{\pi}{4})\vec{i} + 5\cos(\frac{\pi}{4})\vec{j} = \boxed{-\frac{\sqrt{2}}{2}\vec{i} + \frac{5\sqrt{2}}{2}\vec{j}}$

$r''(t) = -\cos t\,\vec{i} - 5\sin(t)\vec{j}$

$r''(\frac{\pi}{4}) = -\cos(\frac{\pi}{4})\vec{i} - 5\sin(\frac{\pi}{4})\vec{j} = \boxed{-\frac{\sqrt{2}}{2}\vec{i} + -\frac{5\sqrt{2}}{2}}$

$\rightarrow$

$\vec{r} = \vec{r_0} + \vec{V_0}\Delta t$

$\Downarrow$

8b.) If the particle were to fly off (in a straight line) the wire at $t = \frac{\pi}{4}$, where would the particle be at $t = \frac{\pi}{2}$

initial position: $\vec{r}(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle$

initial velocity: $\vec{v}(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle$

$\Delta t = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

·Thus at $t = \frac{\pi}{2}$ the particle is at

$\vec{r}(\frac{\pi}{4}) + (\frac{\pi}{2} - \frac{\pi}{4}) v(\frac{\pi}{4})$

$= \boxed{\langle \frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle + \frac{\pi}{4} \langle -\frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle}$

·note· if it helps think of this problem like a physics 1 problem, where something is at an initial position at $r(\frac{\pi}{4})$. Since you want to find the new position at $t = \frac{\pi}{2}$ you simply add your initial position and the distance you moved in the Δt

$\boxed{x_f = x_0 + vt}$ $vt = x$